A Note on Hamilton Cycles in Kneser Graphs

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Abstract

The Kneser graph $K(n,k)$ has as vertices the k-subsets of $\{1,2,...,n\}$ where two vertices are adjacent if the k-subsets are disjoint. In this paper we use a computational heuristic of Shields and Savage to extend previous results and show that all connected Kneser graphs (except the Petersen graph) have Hamilton cycles when $n \leq 27$. A similar result is shown for bipartite Kneser graphs.

1 Introduction

In this paper we use a computational heuristic of the authors to extend the known results for Hamilton cycles in Kneser graphs, $K(n,k)$, and bipartite Kneser graphs, $H(n,k)$. With the exception of the Petersen graph, $K(5,2)$, these have long been conjectured to have Hamilton cycles for $n > 2k$, but neither a constructive nor an existential proof is known. For $n > 2k$, both $K(n,k)$ and $H(n,k)$ are connected and vertex transitive, so the nonexistence of a Hamilton path in $K(n,k)$ or $H(n,k)$ for some n,k would provide a counterexample to the Lovász conjecture [12] that every connected, undirected, vertex transitive graph has a Hamilton path.

The *Kneser graph* $K(n,k)$ has as vertices the k-subsets of $\{1,2,...,n\}$, where two vertices are adjacent if the k-subsets are disjoint. A related

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graph, the *uniform subset graph* $G(n,k,t)$, also has as vertices the k-subsets of $\{1,2,...,n\}$ but here two vertices are adjacent if their intersection has cardinality t. So $K(n,k)=G(n,k,0)$. The *bipartite Kneser graph* $H(n,k)$ has as its partite sets the k- and $(n-k)$-subsets of $\{1,2,\ldots,n\}$, respectively. Two vertices from different partite sets are adjacent iff one is a subset of the other.

There is a one-to-one correspondence between the k-subsets of $\{1,2,...,n\}$ and the set of n-bit binary numbers with exactly k ones and $n-k$ zeros, which simplifies the computer representation and manipulation of these graphs. The graph $K(n,k)$ has $\binom{n}{k}$ vertices while $H(n,k)$ has twice as many vertices and both graphs are regular of degree $\binom{n-k}{k}$.

Simpson [19] studied bipartite graphs $B(G)$ constructed from a graph G with vertex set $V(G)=\{x_1,x_2,\cdots,x_n\}$. The vertex set of $B(G)$ consists of two copies of $V(G)$ denoted $S=\{y_1,y_2,\cdots,y_n\}$ and $T=\{z_1,z_2,\cdots,z_n\}$ and (y_i,z_j) is an edge in $B(G)$ iff (x_i,x_j) is an edge in G. He showed that $H(n,k)=B(K(n,k))$.

Chen and Lih [3] showed that $G(n,k,t)$, and therefore $K(n,k)$, are both vertex transitive and edge transitive. Simpson showed that if G has no loops and is vertex (edge) transitive then $B(G)$ is also vertex (edge) transitive and therefore $H(n,k)$ is both vertex and edge transitive.

The graph $K(5,2)$ is the Petersen graph, which has a Hamilton path but not a Hamilton cycle, although the bipartite graph $H(5,2)$ does have a Hamilton cycle. It has long been conjectured that $K(n,k)$ and $H(n,k)$ (with the exception of $K(5,2)$) have Hamilton cycles when $n>2k$.

In [18], we developed a polynomial-time Hamilton path heuristic which is a refinement of a technique of Pósa [16]. Given a graph G and a vertex v_0 of G, Pósa's algorithm constructs a path iteratively, starting from v_0, adding (if possible) in the ith iteration a new vertex v_i adjacent to v_{i-1}. If this is not possible, and if v_j is a neighbor of v_{i-1} for some $j<i-2$, a *rotation* is performed on a section of the path $v_0,v_1,\ldots,v_{i-1}$ to obtain a path $v_0,v_1,\ldots,v_j,v_{i-1},v_{i-2}\ldots v_{j+1}$. The hope is that v_{j+1} will have a neighbor not yet on the path and the path can be extended.

The heuristic in [18] improves the chance for extension by building a breadth-first search tree to consider sequences of possible rotations and the new endpoints which would result. Tree building stops when an endpoint is found from which the path can be extended, at which time a sequence of one or more rotations is performed and the extension process continues. If there is no such endpoint, the algorithm terminates without success. However, if G has a Hamilton path, and if we get lucky, the algorithm finds a Hamilton path. With minor modification, we can use this heuristic for Hamilton cycles as well.

The algorithm performs surprisingly well on several classes of graphs.

We tested it on the middle two levels graphs $H(2k+1, k)$ in [18] and showed Hamilton cycles for $1 \leq k \leq 15$. Several enhancements were required to be able to run on the very large graphs in this class, but the basic algorithm was the same. In this paper, we use this program, with additional modifications described in Section 4, to complete the verification of the hamiltonicity of all the $K(n, k)$ (except for the Peterson graph) and $H(n, k)$ graphs for $n \leq 27$.

In Section 2 we describe the previous work and our results for Kneser graphs, $K(n, k)$. In Section 3 we extend these results to the corresponding bipartite Kneser graphs $H(n, k)$ using a result due to Simpson and results from our prior work. We show that the Simpson technique cannot be used when $n = 2k + 1$. In Section 4 we discuss algorithmic tradeoffs used in reaching these results.

2 Kneser graphs

The Kneser graph $K(2k-1, k-1)$ is also known as the odd graph O_k. O_2 is a triangle, which has a Hamilton cycle, while O_3 is the Petersen graph, which has no Hamilton cycle. Balaban [1] studied the odd graph as the "k-valent halved combination graph" and exhibited Hamilton cycles for $k = 4$ and $k = 5$. Meredith and Lloyd [14, 15] established Hamilton cycles in O_k for $k = 6$ and $k = 7$ and Mather [13] showed a Hamilton cycle for $k = 8$.

Heinrich and Wallis [10] showed that infinitely many of the Kneser graphs, $K(n, k)$ have Hamilton cycles. In particular, $K(n, k)$ has a Hamilton cycle for $k = 1$, $n \geq 3$ (the complete graph K_n), for $k = 2$, $n \geq 6$ and for $k = 3$, $n \geq 7$. Using a theorem of Baranyai [2] they derived the more general result that $K(n, k)$ has a Hamilton cycle for

$$n \geq k + \frac{k\sqrt[k]{2}}{\sqrt[k]{2} - 1}$$

which tends asymptotically to $k + k^2/\log_2 e$.

The circumference of a graph is the length of its longest cycle. Chen and Lih [3] studied the uniform subset graphs $G(n, k, t)$ and obtained results about the circumference of such graphs, from which they proved that $K(n, k)$ has a Hamilton cycle when $n \geq e(k)$ where

$$e(k) = \min\{n | n > 2k \text{ and } \tbinom{n-1}{k-1}/\tbinom{n-k}{k} \leq 1\}.$$

This implies that $K(n, k)$ has a Hamilton cycle for $n \geq (1+o(1))k^2/\log k$ [7].

Chen [6] used Baranyai's theorem [2] again to show that $K(n, k)$ has a Hamilton cycle for $n \geq 3k$, a dramatic improvement. She [4] subsequently improved this to show that $K(n, k)$ has a Hamilton cycle whenever

$$n \geq \frac{3k + 1 + \sqrt{5k^2 - 2k + 1}}{2}.$$

	$K(n,k)$ $k < n/2$												
n	1	2	3	4	5	6	7	8	9	10	11	12	13
3	Δ												
4	H												
5	H	P											
6	H	H											
7	H	H	A										
8	H	H	H										
9	H	H	H	A									
10	H	H	C_2	C_1									
11	H	H	H	C_1	L								
12	H	H	H	C_2	S								
13	H	H	H	C_2	S	L							
14	H	H	H	C_2	S	S							
15	H	H	H	C_2	C_2	S	M						
16	H	H	H	B	C_2	S	S						
17	H	H	H	B	C_2	S	S	S					
18	H	H	H	B	C_2	C_2	S	S					
19	H	H	H	B	C_2	C_2	S	S	S				
20	H	H	H	B	C_2	C_2	S	S	S				
21	H	H	H	B	C_2	C_2	C_2	S	S	S			
22	H	H	H	B	B	C_2	C_2	C_3	S	S			
23	H	H	H	B	B	C_2	C_2	C_3	S	S	S		
24	H	H	H	B	B	C_2	C_2	C_2	S	S	S		
25	H	H	H	B	B	C_2	C_2	C_2	C_3	S	S	S	
26	H	H	H	B	B	C_2	C_2	C_2	C_3	S	S	S	
27	H	H	H	B	B	C_2	C_2	C_2	C_2	C_3	S	S	S

Δ - Triangle graph
P - Petersen Graph
A - Balaban [1]
L - Meredith & Lloyd [14]
M - Mather [13]
H - Heinrich & Wallis [10]
B - Chen & Lih [3]
C_1 - Chen [5]
C_2 - Chen [6]
C_3 - Chen [4]
S - Current Results

Table 1: Known results for connected Kneser graphs $K(n,k)$ for $n \leq 27$.

We have summarized the known results for Kneser graphs $K(n,k)$ with $n \leq 27$ in Table 1. The previous work leaves gaps when

$$2k+1 < n < \frac{3k+1+\sqrt{5k^2-2k+1}}{2} \qquad (k = 5,6,7)$$

and

$$2k+1 \leq n < \frac{3k+1+\sqrt{5k^2-2k+1}}{2} \qquad (k \geq 8).$$

Our Hamilton cycle program was able to find Hamilton cycles in all of the graphs in Table 1, thus filling in these gaps.

3 Bipartite Kneser graphs

The bipartite Kneser graph $H(n,k)$ has also been studied under several names. Dejter, Cordova and Quintana [9] constructed Hamilton cycles in $H(16,7)$ and $H(19,9)$ and cited earlier work [8] in which Hamilton cycles had been found using similar methods in $H(2k+1,k)$ for $k \leq 8$, in $H(2k+2,k)$ for $k \leq 6$ and in $H(6,3)$. Simpson [19] showed that if $|G|$ is odd and G has a Hamilton cycle then so does $B(G)$. If $|G|$ is even and $C = x_1, x_2, \cdots, x_n, x_1$ is a cycle in G then $B(G)$ has a Hamilton cycle if there is (i) a vertex x_i with i odd, adjacent to x_1 and (ii) x_n is adjacent to either $x_i - 1$ or $x_i + 1$. He showed that $H(n,k)$ has a Hamilton cycle when

$$2\binom{n-1}{k-1} \leq 1 + \binom{n-k}{k}.$$

A slightly weaker condition was given in [20] where it was shown that $H(n,k)$ has a Hamilton cycle when $n \geq (3k^2+k+2)/2$.

Hurlbert [11] also studied $H(n,k)$ calling it the antipodal layers problem and he showed that $H(n,k)$ has a Hamilton cycle for $n > ck^2 + k$ for large enough k.

Chen [6] showed that $H(n,k)$ has a Hamilton cycle for $n > 3k$, and $H(3k,k)$ has one when $\binom{3k}{k}$ is odd, by extending her results for Kneser graphs in a manner similar to that used by Simpson for extending the results for a general graph G to the bipartite graph $B(G)$. She [4] subsequently improved this to show that $H(n,k)$ has a Hamilton cycle whenever

$$n \geq \frac{3k+1+\sqrt{5k^2-2k+1}}{2}.$$

The special case of $H(2k+1,k)$ is also known as the *middle two levels problem* from its relationship to the middle two levels of the Boolean lattice. The current best result, due to the present authors [18] is that $H(2k+1,k)$ has a Hamilton cycle for all $k \leq 15$. Earlier results are described in that paper.

We have summarized the known results for bipartite Kneser graphs $H(n,k)$ with $n \leq 27$ in Table 2. The previous work leaves gaps when

$$2k+2 < n < \frac{3k+1+\sqrt{5k^2-2k+1}}{2} \ (k = 4,5,6,7)$$

and

$$2k+1 < n < \frac{3k+1+\sqrt{5k^2-2k+1}}{2} \ (k \geq 8).$$

We were able to fill in all of these gaps using our Hamilton cycle program. Although the heuristic could be run independently for Kneser and bipartite Kneser graphs, instead we added code to perform the Simpson [19]

	$K(n,k)\ k < n/2$												
n	1	2	3	4	5	6	7	8	9	10	11	12	13
3	D_1												
4	D_1												
5	I	D_1											
6	I	D_1											
7	I	H	D_1										
8	I	I	D_1										
9	I	I	D_1	D_1									
10	I	I	C_2	D_1									
11	I	I	H	S	D_1								
12	I	I	H	C_2	D_1								
13	I	I	C_2	C_2	S	D_1							
14	I	I	H	C_2	C_3	D_1							
15	I	I	H	C_2	C_2	S	D_1						
16	I	I	I	C_2	C_2	C_3	D_2						
17	I	I	I	C_2	C_2	C_3	S	D_1					
18	I	I	I	C_2	C_2	C_3	S	S					
19	I	I	I	C_2	C_2	C_2	C_3	S	D_2				
20	I	I	I	H	C_2	C_2	C_3	S	S				
21	I	I	I	H	C_2	C_2	C_3	S	S	M			
22	I	I	I	H	C_2	C_2	C_2	C_3	S	S			
23	I	I	I	H	C_2	C_2	C_2	C_3	S	S	M		
24	I	I	I	H	C_2	C_2	C_2	C_2	C_3	S	S		
25	I	I	I	H	C_2	C_2	C_2	C_2	C_3	S	S	S_1	
26	I	I	I	H	C_2	C_2	C_2	C_2	C_3	S	S	S	
27	I	I	I	I	C_2	C_2	C_2	C_2	C_2	C_3	S	S	S_1

D_1 - Dejter [8]
D_2 - Dejter et al [9]
I - Simpson
H - Hurlbert [11]
M - Moews & Reid [17]
S_1 - Shields & Savage [18]
C_2 - Chen [6]
C_3 - Chen [4]
S - Current Results

Table 2: Known results for connected bipartite Kneser graphs $H(n,k)$ for $n \leq 27$.

test whenever a Hamilton cycle was found in a Kneser graph under test. This provided a relatively quick check to determine if the bipartite analog, $H(n,k)$, of a Kneser graph, $K(n,k)$, was Hamiltonian. The cases in which this test did not prove $H(n,k)$ to be Hamiltonian when $K(n,k)$ was were all cases where $n = 2k+1$ or, in other words, $H(n,k)$ was an instance of the middle two levels problem. In all of these cases the authors had previously shown $H(n,k)$ to be Hamiltonian [18].

Since the cases where the Simpson test failed represented just one of many possible Hamilton cycles in $K(n,k)$ the question arises as to whether a Hamilton cycle in $K(n,k)$ does exist for which the Simpson test would work. We show this is not possible.

Theorem 1 *If $n = 2k + 1$ then no Hamilton cycle in $K(n, k)$ can satisfy the Simpson test.*

Proof. The Simpson test requires that four vertices of $K(n, k)$ form a cycle. If $k = 1$ then $|K(n, k)| = 3$ and no 4-cycle exists. Suppose $k > 1$ and that four vertices, a, b, c, d of $K(n, k)$ form a cycle. Now a, b, c, d are $k-$subsets of $\{1, 2, ..., n\}$. Since $n = 2k+1$ there exists $i \in \{1, 2, ..., n\}$ such that $i \notin a$ and $i \notin b$. Hence $i \in c$ otherwise $c = a$. Since $c \cap d = \phi$ we have $i \notin d$. Since $n = 2k + 1$ this is impossible since $d \neq b$. 2

Thus it can be seen that the Simpson test may fail to show a graph $B(G)$ to be Hamiltonian even when a Hamilton cycle in the underlying graph G is known.

4 The heuristic and its performance

As described in Section 1, the heuristic we used to find Hamilton cycles in the Kneser graphs is the rotation-extension heuristic of Shields and Savage [18]. $K(27, 13)$ has 20,058,300 vertices, each of degree 14, making 140,408,100 edges. $K(27, 10)$ has 8,436,285 vertices, each of degree 19,448, making over $113 * 10^9$ edges.

Handling such large graphs required improvements in both the space and time requirements for the heuristic. In a system where the program requires more memory than the system has available, memory is swapped to disk by the operating system. When this occurred, execution times tended to increase significantly.

Some optimization was done initially to reduce storage requirements. Part of this was later taken back to significantly reduce execution time in the tree building phase. When storage again became a problem with large graphs of high degree, another tradeoff was required. Even if virtual memory was large enough to hold all adjacency lists, keeping them resulted in significant memory swapping, particularly during the tree building phase. Instead, we used some CPU cycles to generate adjacency lists as required. The significant saving in swapping more than offset the cost of the extra computations for these problems. These improvements allowed us to obtain these results on a 1.6GHz personal computer with 640MB of memory.

The long running times required for larger graphs also made a checkpoint function desirable to allow restarting a problem part way through. To accomplish this we wrote a file containing the partial path along with some key graph information every half hour.

We have summarized the running times and number of rotation operations needed in Table 3. Note that the times shown here include (i) the time to test hamiltonicity of $K(n, k)$, (ii) the time to take a checkpoint every half hour and (iii) the time required to perform the Simpson test on all possible rotations of the cycle in $K(n,k)$ to determine if $H(n,k)$ has a Hamilton cycle.

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n	k	$\|V\|$	Deg	Time (sec)	Rotations
10	4	210	15	0	16
11	4	330	35	0	10
11	5	462	6	0	49
12	4	495	70	0	5
12	5	792	21	0	34
13	5	1287	56	0	33
13	6	1716	7	0	201
14	5	2002	126	0	16
14	6	3003	28	0	90
15	5	3003	252	0	11
15	6	5005	84	0	64
15	7	6435	8	0	1178
16	6	8008	210	0	47
16	7	11440	36	1	288
17	6	12376	462	1	36
17	7	19448	120	0	181
17	8	24310	9	1	4020
18	6	18564	924	3	39
18	7	31824	330	3	142
18	8	43758	45	1	947
19	7	50388	792	11	87
19	8	75582	165	6	506
19	9	92378	10	31	14628
20	7	77520	1716	33	57
20	8	125970	495	24	303
20	9	167960	55	39	2932
21	7	116280	3432	84	36
21	8	203490	1287	82	192
21	9	293930	220	64	1493
21	10	352716	11	711	52336
22	8	319770	3003	309	139
22	9	497420	715	191	802
22	10	646646	66	582	10081
23	8	490314	6435	750	108
23	9	817190	2002	570	493
23	10	1144066	286	611	4457
23	11	1352078	12	10753	94270
24	8	735471	12870	2069	73
24	9	1307504	5005	1968	315
24	10	1961256	1001	1316	2199
24	11	2496144	78	8206	33051
25	9	2042975	11440	6171	228
25	10	3268760	3003	3699	1212
25	11	4457400	364	6155	13291
25	12	5200300	13	159817	344435
26	9	3124550	24310	19032	145
26	10	5311735	8008	15137	765
26	11	7726160	1365	9799	6164
26	12	9657700	91	106054	111573
27	10	8436285	19448	52759	583
27	11	13037895	4368	29192	3361
27	12	17383860	455	81878	42059
27	13	20058300	14	2476824	1299446

Table 3: Times and number of rotations required to determine if the Kneser graph $K(n,k)$ and bipartite Kneser graph $H(n,k)$ are Hamiltonian

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